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ENDOMORPHISMS OF A LEBESGUE SPACE III

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ABSTRACT

A new invariant is introduced for regular isomorphisms, which are isomorphisms by codes that anticipate a finite amount of the future. With the help of this invariant it is shown that the Bernoulli automorphism (p, q) is not regularly isomorphic to the Markov automorphism $\begin{pmatrix} pq \\ qp \end{pmatrix}$, $p \neq q$, and that neither of these is regularly isomorphic to the Markov automorphism $\begin{pmatrix} qp \\ pq \end{pmatrix}$.

0. Introduction

This paper is a continuation of [1] and [2]. In [1] Peter Walters and the author investigated the isomorphism problem for endomorphisms of a Lebesgue space from the point of view of faithful coding without anticipation. In [2] Robin Fellgett and the author modified this point of view so that anticipatory codes between automorphisms were allowed if they anticipated only a finite amount of the future. Such codes are called *regular isomorphisms.* The purpose of this paper is to introduce a new regular isomorphism invariant. Concretely we are able to show that the Bernoulli automorphism $\begin{pmatrix} pq \\ pq \end{pmatrix}$ is *not* regularly isomorphic to the Markov automorphism $\begin{pmatrix} pq \\ qp \end{pmatrix}$ $(p \neq q)$, and neither of these is regularly isomorphic to the Markov automorphism $\begin{pmatrix} qp \\ pq \end{pmatrix}$.

The associated endomorphisms are not isomorphic, as has been shown in [3], [4], [5] and [1]. That the automorphisms are isomorphic (without qualification) follows from [6], since they have the same entropy. In other words, the substance of our concrete result is that any faithful code between two of these examples must anticipate unbounded amounts of the future.

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1. Definitions

Let T be an endomorphism of the Lebesgue space $(X, \mathscr{B}, m)$ and let $\hat{T}$ be its natural extension to the Lebesgue space $(\hat{X}, \hat{\mathscr{B}}, \hat{m})$, so that we have a commutative diagram:

$$\begin{array}{ccc} & \hat{T} & \\ & \hat{X} \to \hat{X} & \\ \pi \downarrow & & \downarrow \pi \\ & X \to X & \\ & T & \end{array}$$

and $\bar{\mathscr{B}} = \pi^{-1}\mathscr{B}$ has the properties: $\hat{T}^{-1}\bar{\mathscr{B}} \subset \bar{\mathscr{B}}$, $\hat{T}^n\bar{\mathscr{B}} \uparrow \hat{\mathscr{B}}$ (c.f. [7]). With this set up we call $(\hat{X}, \hat{\mathscr{B}}, \bar{\mathscr{B}}, \hat{m}, \hat{T})$ a process and $\bar{\mathscr{B}}$ is called its *future.*

Two processes $(\hat{X}_i, \hat{\mathscr{B}}_i, \bar{\mathscr{B}}_i, \hat{m}, \hat{T}_i)$ are said to be *regularly isomorphic* if there is an isomorphism ϕ $(\phi\hat{T}_1 = T_2\phi)$ such that $\phi^{-1}\bar{\mathscr{B}}_2 \subset \hat{T}_1^k\bar{\mathscr{B}}_1$, $\phi\bar{\mathscr{B}}_i \subset \hat{T}_2^k\bar{\mathscr{B}}_2$ for some $k \geqq 0$. Two endomorphisms $(X_i, \mathscr{B}_i, m_i, T_i)$ are said to be *shift* equivalent if there exist homomorphisms ϕ, ψ with $\phi T_1 = T_2\phi$, $\psi T_2 = T_1\psi$ and $\psi\phi = T_1^k$, $\phi\psi = T_2^k$ for some $k \geqq 0$. (This notion was introduced by Williams in [8], [9] for topological and algebraic categories.)

In [2] the following proposition was proved which enables us to dispense with automorphisms, when investigating regular isomorphy.

Proposition 1. *Two processes are regularly isomorphic if and only if their associated endomorphisms are shift equivalent.*

If $\mathscr{A}$, $\mathscr{C}$ are two sub-σ-algebras of $\mathscr{B}$, $I(\mathscr{A}/\mathscr{C})$ will denote the conditional information of $\mathscr{A}$ given $\mathscr{C}$ and we shall refer to $I(\mathscr{B}/T^{-1}\mathscr{B}) = I_T$ as the information function of $(X, \mathscr{B}, m, T)$.

If T_1, T_2 are shift equivalent by ϕ, ψ then

$$I(\mathscr{B}_1/T_1^{-1}\mathscr{B}_1) = I(\mathscr{B}_2/T_2^{-1}\mathscr{B}_2)\circ\phi + gT_1 - g, \tag{1.1}$$

where $g = I(\mathscr{B}_1/\phi^{-1}\mathscr{B}_2)$. This relationship can be expressed by saying that the two information functions are cohomologous. (Functions of the form $gT_1 - g$ are additive T_1 coboundaries.) The main aim of [2] was to exploit this relationship via a numerical invariant. Here we introduce a group invariant which, happily, is frequently computable and often non-trivial. The point to be stressed is that any other *canonical* function satisfying (1.1) would serve us in place of the information function. (But what canonical functions are there?)

If T is an endomorphism of $(X, \mathscr{B}, m)$ we can define an infinite measure preserving transformation T' on $X \times R$ called the *canonical line extension* of T by

$$T'(x, y) = (Tx, y + I_T(x) - \int I_T dm).$$

We can also define *canonical circle extensions* on $X \times K$ (where $K = \{z : |z| = 1\}$) by $T'(x, y) = (Tx, z \cdot \exp 2\pi i[c + dI_T(x)])$ for each $c, d \in R$.

Using (1.1) it is not difficult to prove:

PROPOSITION 2. *If T_1, T_2 are shift equivalent endomorphisms then their line extensions are shift equivalent and their circle extensions (for each $c, d \in R$) are shift equivalent.*

The purpose of this proposition is, that by distinguishing between extensions one automatically distinguishes between the base endomorphisms. We have not so far been able to employ line extensions. Circle extensions, however, are implicitly used in the remainder of this paper, through the following invariant:

Let $\Lambda(T) = \{(c,d) \in R \times R : \exp 2\pi i(c + dI_T) = F(T)/F$ for some measurable $F : X \to K\}$. In this definition I_T may be replaced by any other function additively cohomologous to it. Clearly $\Lambda(T)$ is a sub-group of $R \times R$.

THEOREM 1. *If T_1, T_2 are shift equivalent and $I(\mathcal{B}_1/T_1^{-1}\mathcal{B}_1)$ is finite a.e. then $\Lambda(T_1) \equiv \Lambda(T_2)$.*

This is a simple application of (1.1).

2. Computations of $\Lambda(T)$

(2.1) *β transformations*

Let $Tx = \beta x \bmod 1$ where $\beta > 1$. There is a T invariant probability measure m equivalent to Lebesgue measure l on $[0, 1)$ such that dm/dl is bounded and bounded away from zero [9], [10], [11]. It is not difficult to show that $I(\mathcal{B}/T^{-1}\mathcal{B})$ is cohomologous to $\log dlT/dl \equiv \log \beta$. Hence

$$\Lambda(T) = \left\{\{(c, d) : \exp 2\pi i(c + d \log \beta) = \frac{F(T)}{F}\right\}.$$

Since T is weak-mixing (in fact it is an exact endomorphism)

$$\Lambda(T) = \{(c, d) : c + d \log \beta \in Z\} = \{(m - d \log \beta, d) : m \in Z,\ d \in R\}.$$

(2.2) *Markov endomorphisms*

Let T be a Markov endomorphism defined by the transition matrix P and stable initial vector $p\,(pP = p)$ where $\Sigma p(i) = 1$. Then

$$I(\mathcal{B}/T^{-1}\mathcal{B})(x) = \log\left[\frac{p(j)}{p(i)P(i,j)}\right] \quad \text{when } x_0 = i,\ x_1 = j.$$

This function is cohomologous to $J(x) = -\log P(i,j)$, where $x_0 = i$, $x_1 = j$. Hence

$$\Lambda(T) = \{(c,d) : \exp 2\pi i[c - d \log P(i,j)] = F(j)/F(i)$$

for some F mapping the states to $K\}$.

This latter remark depends on the following:

Proposition 3. *If T is a Markov endomorphism and if $F: X \to K$ is such that FT/F depends on the first $m+1$ variables $x_0, x_1, \cdots x_m$ only, then F depends on only the first m variables $x_0, x_1, \cdots x_{m-1}$.*

(2.3) *Bernoulli endomorphisms*

As a special case of (2.2) we see that if T is a Bernoulli endomorphism defined by the probability vector p, then

$$\Lambda(T) = \{(c,d): \exp 2\pi i\,[c - d \log p(i)] = 1 \quad \text{for all } i\}.$$

Let T be based on $(\frac{1}{2}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$ and let S be based on $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. ($\hat{T}, \hat{S}$ were proved to be isomorphic by Meshalkin. In [2] it was shown that they are not regularly isomorphic. Here we present another proof.)

$$\Lambda(T) = \{(c,d) : c + d \log 2 \in Z,\ c + d \log 8 \in Z\}.$$

Hence $d = m/2 \log 2$, $m \in Z$, $c = n/2$, $n \in Z$, where $m + n$ is even, i.e.

$$\Lambda(T) = \left\{\left(\frac{n}{2}, \frac{m}{2\log 2}\right): m + n \ \text{ is even}, m, n \in Z\right\}.$$

$$\Lambda(S) = \{(m - d \log 4, d):\ m \in Z,\ d \in R\}.$$

Thus $\Lambda(T) \neq \Lambda(S)$, so that S, T are not shift equivalent. We note also that if the $p(i)$ are not all identical, then $\Lambda(T)$ for a Bernoulli endomorphism is countable, so that natural extensions of β transformations are not in general regularly isomorphic to Bernoulli automorphisms. This was also proved and discussed in [2].

(2.4) $\begin{pmatrix} pq \\ pq \end{pmatrix}, \begin{pmatrix} pq \\ qp \end{pmatrix}, \begin{pmatrix} qp \\ pq \end{pmatrix}, \ p \neq q$

Let T_1, T_2, T_3 be the Markov endomorphisms with the above transition matrices. T_1 is Bernoulli so that

$$\Lambda(T_1) = \{(c,d): c - d \log p \in Z, c - d \log q \in Z\}.$$

In other words, we require for $m, n \in Z$

$$\begin{pmatrix} 1 - \log p \\ 1 - \log q \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} m \\ n \end{pmatrix}$$

$$\text{i.e. } \begin{pmatrix} c \\ d \end{pmatrix} = \frac{1}{\log p/q} \begin{pmatrix} -\log q & \log p \\ -1 & 1 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix}.$$

Hence

$$\Lambda(T_1) = \left\{ \left(\frac{-m \log q + n \log p}{\log (p/q)}, \frac{n-m}{\log (p/q)} \right) : \ m, n \in Z \right\}.$$

T_2 is a Markov endomorphism, so we need a function F mapping states to K such that

$$\exp 2\pi i\,(c - d \log p) = \frac{F(1)}{F(1)} = \frac{F(2)}{F(2)}$$

and

$$\exp 2\pi i\,(c - d \log q) = \frac{F(2)}{F(1)} = \frac{F(1)}{F(2)}.$$

Hence $F(2) = \pm F(1)$. Taking $F(2) = F(1)$, we get $\Lambda(T_2) \supset \Lambda(T_1)$. Taking $F(2) = -F(1)$, we get $c - d \log q$ is a half integer and $c - d \log p \in Z$. Hence

$$\Lambda(T_2) = \left\{ \left(\frac{-m \log q + \alpha(n/2) \log p}{\log (p/q)}, \frac{(n/2) - m}{\log (p/q)} \right); \quad m, n \in Z \right\}$$

and $\Lambda(T_2) \neq \Lambda(T_1)$.

Interchanging p, q we see that

$$\Lambda(T_3) = \left\{ \left(\frac{-m \log p + (n/2) \log q}{-\log (p/q)}, \frac{(n/2) - m}{-\log (p/q)} \right): \quad m, n \in Z \right\}$$

$$= \left\{ \left(\frac{-(m/2) \log q + n \log p}{\log (p/q)}, \frac{-(m/2) + n}{\log (p/q)} \right): \quad m, n \in Z \right\}.$$

Hence the groups $\Lambda(T_1)$, $\Lambda(T_2)$, $\Lambda(T_3)$ are all distinct and no two of T_1, T_2, T_3 are shift equivalent. In other words, no two of $\hat{T}_1$, $\hat{T}_2$, $\hat{T}_3$ are regularly isomorphic.

3. Compact abelian group extensions of Markov endomorphisms

In conclusion we mention one more application of Proposition 3, which latter depends on the following:

LEMMA. *If T is an endomorphism of $(X,\mathcal{B},m)$ and $T^{-1}\mathcal{A} \subset \mathcal{A} \subset \mathcal{B}$, then either f is measurable with respect to $\mathcal{A}$ or $E\,(f \mid \bigvee_{n=0}^{\infty} T^{-n}\mathcal{A}) = 0$ a.e. whenever fT/f is $\mathcal{A}$ measurable and $|f| = 1$.*

The application we have in mind is the following

THEOREM 2. *Let T be a mixing Markov endomorphism on X and let ϕ be a function of $n+1$ variables to a compact abelian group G, then $S: X \times G \to X \times G$ defined by $S(x,g) = (Tx, \phi(x_0, \cdots, x_n)g)$ is an exact endomorphism if and only if*

$$\frac{F(x_1 \cdots x_n)}{F(x_0 \cdots x_{n-1})} = k\gamma_0\phi(x_0 \cdots x_n), \quad |F| = 1, \ \gamma \in \hat{G}, \ |k| = 1$$

has only the solution $\gamma \equiv 1$. $k = 1$, F constant.

(I am informed by Paul Shields that Roy Adler and he can show that under the above exactness condition $\hat{S}$ is Bernoulli. Ken Thomas also has a proof that $\hat{S}$ is Bernoulli.)

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